

PROCEEDINGS OF THE  
AMERICAN MATHEMATICAL SOCIETY  
Volume 127, Number 9, Pages 2653–2655  
S 0002-9939(99)05423-4  
Article electronically published on May 19, 1999

## EULER CHARACTERISTIC OF THE MILNOR FIBRE OF PLANE SINGULARITIES

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(Communicated by Ron Donagi)

**ABSTRACT.** We give a formula for the Euler characteristic of the Milnor fibre of any analytic function  $f$  of two variables. This formula depends on the intersection multiplicities, the Milnor numbers and the powers of the branches of the germ of the curve defined by  $f$ . The goal of the formula is that it use neither the resolution nor the deformations of  $f$ . Moreover, it can be use for giving an algorithm to compute it.

### 1. INTRODUCTION

In this note we deal with germs of analytic functions  $f$  of two complex variables with  $f(0) = 0$  and its factorization  $f = f_1^{q_1} \cdots f_r^{q_r}$  into irreducible factors, such that  $f_i/f_j, 1 \leq i, j \leq r$ , are as power series not units. Let  $(C, 0)$  be the germ of the plane curve defined by the local equation  $f = 0$  and let  $(C_i, 0), i = 1, \dots, r$ , be its reduced branches defined by  $f_i = 0$ .

The local curve  $C$  defines a *link with multiplicities*  $L := C \cap S_\varepsilon^3$ , in the sphere of radius  $\varepsilon > 0$  around  $0 \in \mathbb{C}^2$ , which does not depend on  $\varepsilon$  provided  $\varepsilon$  is small enough. The link  $L$  consists of the components  $C_i \cap S_\varepsilon^3$ , with multiplicities  $q_i$  and determines the topological type of the germ  $C$ . Moreover, Milnor proved that the map  $\frac{f}{|f|} : S_\varepsilon^3 \setminus L \rightarrow S^1$  is a  $C^\infty$ -locally trivial fibration, the *Milnor fibration*. Any fibre  $F$  of this fibration is called the *Milnor fibre of  $f$*  (see [M, Theorem 4.8]). A'Campo [A] and Eisenbud-Neumann [EN], using different methods, calculated many topological invariants of the fibration  $\frac{f}{|f|}$  from the resolution graph or the splicing diagrams. We are only interested in the Euler characteristic  $\chi(F)$  of the surface  $F$ . If  $f$  is reduced, i.e. every power  $q_i$  is equal to one, the Euler characteristic of  $F$  is  $1 - \mu(C, 0)$ , where  $\mu(C, 0)$  is the Milnor number of the isolated singularity of  $C$ . Moreover the Euler characteristic of  $F$  is related to topological and geometric invariants of its branches by the well-known formula:

$$\chi(F) = -2 \sum_{1 \leq i < j \leq r} (C_i, C_j)_0 + \sum_{i=1}^r (1 - \mu(C_i)),$$

where  $(C_i, C_j)_0$  is the intersection multiplicity of  $C_i$  and  $C_j$  at the origin and  $\mu(C_i)$  is the Milnor number of  $C_i$  at the origin (e.g. see [BK]).

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Received by the editors July 24, 1996 and, in revised form, June 27, 1997.

1991 *Mathematics Subject Classification.* Primary 32S05, 14H20; Secondary 14B05.

*Key words and phrases.* Euler characteristic, Milnor fibration, Milnor fibre.

This work was done under the partial support of CAYCIT PB94-291.

On the other hand, when  $f$  is non-reduced Schrauwen [S] expressed the Euler characteristic of  $F$  in terms of special points of suitable deformations of  $f$ . For calculating  $\chi(F)$  in this case one can use the methods of A'Campo or Eisenbud-Neumann and construct the resolution graph or the splicing diagram.

The aim of this note is to give a closed formula for the Euler characteristic of  $F$  without the construction of these graphs.

For every  $q \in \mathbb{N}^r$  set

$$F^q := \{z \in S_\epsilon : \prod_{1 \leq i \leq r, q_i \neq 0} \left(\frac{f_i}{|f_i|}\right)^{q_i}(z) = 1 \text{ and } f_i(z) \neq 0 \quad \forall i = 1, \dots, r\}.$$

Note that, for  $\epsilon$  small, the surface  $F^q$  is the Milnor fibre of the local curve  $C^q := \{f_1^{q_1} \cdots f_r^{q_r} = 0\}$  if and only if all  $q_i \neq 0$ . If some  $q_i$  are zero, but  $q \neq 0$ , then  $F^q$  is the Milnor fibre of  $\prod_{1 \leq i \leq r, q_i \neq 0} f_i^{q_i}$  with punctures, where the number of punctures equals  $\sum_{1 \leq i, j \leq r, q_i \neq 0, q_j = 0} (C_i, C_j)_0(q_i)$ . For  $q = 0$  the space  $F^q$  is just the complement of the link of the curve  $C$ .

Our generalized and closed formula is:

$$\chi(F^q) = - \sum_{1 \leq i < j \leq r} (C_i, C_j)_0 (q_i + q_j) + \sum_{i=1}^r q_i (1 - \mu(C_i)).$$

I am indebted to the referee for suggesting how to improve the presentation of the proof of the formula.

## 2. PROOF OF THE FORMULA

The formula follows from the two following lemmas.

**Lemma 1.** *The function  $q \in \mathbb{N}^r \rightarrow \chi(F^q)$  is additive.*

*Proof.* Let  $\pi : X \rightarrow \mathbb{C}^2$  be a proper modification of  $\mathbb{C}^2$  above the origin such that, for every point on the divisor  $E := \pi^{-1}(0)$ , the total transform of the  $\bigcup_{1 \leq i \leq r} C_i$  has normal crossing singularities. Let  $\widetilde{C}_i$  be the strict transform of  $C_i$  by  $\pi$  and  $E_\alpha$ ,  $\alpha \in A$ , the components of  $E$ .

First assume  $q \neq 0$ . Put  $f^q = \prod_{1 \leq i \leq r, q_i \neq 0} f_i^{q_i}$ . Observe that  $F^q$  retracts on  $E \setminus \left(\bigcup_{1 \leq j \leq r, q_j = 0} \widetilde{C}_j\right)$ . With the formula of A'Campo we get:

$$\chi(F^q) = \sum_{\alpha \in A} m(f^q, E_\alpha) \chi(\check{E}_\alpha),$$

where  $\check{E}_\alpha := E_\alpha \setminus \left(\bigcup_{\beta \neq \alpha} E_\beta \cup \bigcup_{1 \leq i \leq r} \widetilde{C}_i\right)$ . Then

$$\chi(F^q) = \sum_{\alpha \in A} \sum_{i=1}^r q_i m(f_i, E_\alpha) \chi(\check{E}_\alpha),$$

since  $m(f^q, E_\alpha) = \sum_{1 \leq i \leq r} q_i m(f_i, E_\alpha)$ .

To prove the additivity it remains to observe that  $\chi(F^0) = 0$ . □

Put  $e_i = (0, \dots, 1, \dots, 0)$ . From the additivity we get:

$$\chi(F^q) = \sum_{i=1}^r q_i \chi(F^{e_i}).$$

**Lemma 2.**

$$\chi(F^{e_i}) = - \sum_{\substack{j=1, \dots, r \\ i \neq j}} (C_i, C_j)_0 + (1 - \mu(C_i)).$$

*Proof.* Remember that  $F^{e_i}$  is the Milnor fibre  $F_i$  with  $\sum_{1 \leq j \leq r, j \neq i} (C_i, C_j)_0$  punctures.  $\square$

Remark that Lemma 1 holds for the case where the germs of the curves  $C_i$  are reduced and have no branch in common. Thus, if we assume

1. each  $f_i$  has no multiple components (i.e.  $f_i$  is squarefree) and
2. for  $i, j \in \{1, \dots, r\}$ ,  $i \neq j$ , the germ  $f_i f_j$  has no multiple components,

then we finally get for the Euler characteristic of the Milnor fibre of  $F$  of  $f = f_1^{q_1} \cdots f_s^{q_s}$ ,  $q_i > 0$ , the formula:

$$\chi(F) = - \sum_{1 \leq i < j \leq s} (C_i, C_j)_0 (q_i + q_j) + \sum_{i=1}^s q_i (1 - \mu(C_i, 0)).$$

To have this formula for squarefree factorization is particularly useful for inductive calculations. If  $R$  is a computable ring with  $\text{char}(R) = 0$  and  $f$  is a polynomial in  $R[x, y]$ , then there exists an algorithm that computes a squarefree decomposition of  $f$  in  $R[x, y]$  (see [BWK, Proposition 2.86, Corollary 2.92]). This is also a squarefree decomposition in  $R\{x, y\}$  and one may then compute the intersection multiplicities and the Milnor numbers. I would like to thank Bernd Martin for showing me the implementation of this algorithm using the computer algebra system SINGULAR, [GPS].

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